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Differential operator specializations of noncommutative symmetric functions

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Abstract

Let K be any unital commutative $\mathbb{Q}$-algebra and $z = (z_1, \dots, z_n)$ commutative or noncommutative free variables. Let t be a formal parameter which commutes with z and elements of K. We denote uniformly by $K\langle\langle z\rangle\rangle$ and $K[[t]]\langle\langle z\rangle\rangle$ the formal power series algebras of z over K and $K[[t]]$, respectively. For any $\alpha \geqslant 1$, let $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ be the unital algebra generated by the differential operators of $K\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$ and $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ the group of automorphisms $F_t(z) = z - H_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ with $o(H_t(z)) \geqslant \alpha$ and $H_{t=0}(z) = 0$. First, for any fixed $\alpha \geqslant 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we introduce five sequences of differential operators of $K\langle\langle z\rangle\rangle$ and show that their generating functions form an $\mathcal{N}CS$ (noncommutative symmetric) system [W. Zhao, Noncommutative symmetric systems over associative algebras, J. Pure Appl. Algebra 210 (2) (2007) 363–382] over the differential algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Consequently, by the universal property of the $\mathcal{N}CS$ system formed by the generating functions of certain NCSFs (noncommutative symmetric functions) first introduced in [I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (2) (1995) 218–348, MR1327096; see also hep-th/9407124], we obtain a family of Hopf algebra homomorphisms $\mathcal{S}_{F_t}: \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$), which are also grading-preserving when F_t satisfies certain conditions. Note that the homomorphisms $\mathcal{S}_{F_t}$ above can also be viewed as specializations of NCSFs by the differential operators of $K\langle\langle z\rangle\rangle$. Secondly, we show that, in both commutative and noncommutative cases, this family $\mathcal{S}_{F_t}$ (with all $n \geqslant 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) of differential operator specializations can distinguish any two different NCSFs. Some connections of the results above with the quasi-symmetric functions [I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, in: Contemp. Math., vol. 34, 1984, pp. 289–301, MR0777705; C. Malvenuto, C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, J. Algebra 177 (3) (1995) 967–982, MR1358493; Richard P. Stanley, Enumerative Combinatorics II, Cambridge University Press, 1999] are also discussed.

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1. Introduction

Let K be any unital commutative $\mathbb{Q}$-algebra and A a unital associative but not necessarily commutative K-algebra. Let t be a formal central parameter, i.e. it commutes with all elements of A, and $A[\![t]\!]$ the K-algebra of formal power series in t with coefficients in A. An $\mathcal{N}$*CS system* over A (see Definition 2.1) by definition is a 5-tuple $\Omega \in A[\![t]\!]^{\times 5}$ which satisfies the defining equations (see Eqs. (2.1)–(2.5)) of the NCSFs (noncommutative symmetric functions) first introduced and studied in the seminal paper [10]. When the base algebra K is clear in the context, the ordered pair (A, Ω) is also called an $\mathcal{N}$*CS system*. In some sense, an $\mathcal{N}$CS system over an associative K-algebra can be viewed as a system of analogs in A of the NCSFs defined by Eqs. (2.1)–(2.5). For some general discussions on the $\mathcal{N}$CS systems, see [29]. For an $\mathcal{N}$CS system over the Grossman–Larson Hopf algebra [9,12] of labeled rooted trees, see [31]. For more studies on NCSFs, see [3,4,14–16,23]. One immediate but probably the most important example of the $\mathcal{N}$CS systems is $(\mathcal{N}Sym, \Pi)$ formed by the generating functions of the NCSFs defined in [10] by Eqs. (2.1)–(2.5) over the free K-algebra $\mathcal{N}Sym$ of NCSFs (see Section 2). It serves as the universal $\mathcal{N}$CS system over all associative K-algebras (see Theorem 2.5). More precisely, for any $\mathcal{N}$CS system (A, Ω), there exists a unique K-algebra homomorphism $\mathcal{S} : \mathcal{N}Sym \to A$ such that $\mathcal{S} : \mathcal{N}Sym \to A$ such that $\mathcal{S}^{\times 5}(\Pi) = \Omega$ (here we have extended the homomorphism $\mathcal{S}$ to $\mathcal{S} : \mathcal{N}Sym[\![t]\!] \to A[\![t]\!]$ by the base extension).

The universal property of the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$ can be applied as follows when an $\mathcal{N}$CS system (A, Ω) is given. Note that, as an important topic in the symmetric function theory, the relations or polynomial identities among various NCSFs have been worked out explicitly (see [10]). When we apply the K-algebra homomorphism $\mathcal{S} : \mathcal{N}Sym \to A$ guaranteed by the universal property of the system $(\mathcal{N}Sym, \Pi)$ to these identities, they are transformed into identities among the corresponding elements of A in the system Ω. This will be a very effective way to obtain identities for certain elements of A if we can show they are involved in an $\mathcal{N}$CS system over A. On the other hand, if an $\mathcal{N}$CS system (A, Ω) has already been well understood, the K-algebra homomorphism $\mathcal{S} : \mathcal{N}Sym \to A$ in turn provides a *specialization* or *realization* [10,22] of NCSFs, which may provide some new understandings on NCSFs. For more studies on the specializations of NCSFs, see the references quoted above for NCSFs.

In this paper, motivated by the studies on the deformations of formal analytic maps of affine spaces in [27,28], we first construct a family of $\mathcal{N}$CS systems over differential operator algebras and then study some properties of the resulted specializations of NCSFs. To be more precise, let us first fix the following notations. Let $z = (z_1, z_2, \ldots, z_n)$ be commutative or noncommutative free variables and t a central parameter, i.e. t commutes with z. To keep notation simple, we use the notations for noncommutative variables uniformly for both commutative and noncommutative variables z. Let $K\langle\langle z\rangle\rangle$ (respectively $K\langle z\rangle$) be the algebra of formal power series (polynomials) in z over K. For any $\alpha \geqslant 1$, let $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ (respectively $\mathcal{D}^{[\alpha]}\langle z\rangle$) be the unital algebra generated by the differential operators of $K\langle\langle z\rangle\rangle$ (respectively $K\langle z\rangle$) which increase the degree in z by at least $\alpha - 1$ and $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ the group of automorphisms $F_t(z) = z - H_t(z)$ of $K[\![t]\!]\langle\langle z\rangle\rangle$ with $o(H_t(z)) \geqslant \alpha$ and $H_{t=0}(z) = 0$. First, for any fixed $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we consider the differential operators, which are the differential operators involved in the Taylor series expansions of $u(F_t(z))$ and $u(F_t^{-1}(z))$ ($u(z) \in K\langle\langle z\rangle\rangle$), the differential operators directly related with the D-Log (see [5–7,20,25] and [26] for the commutative case) of F_t and, finally, two sequences of differential operators that appeared in [28] in the study of deformations of the automorphisms of $K\langle\langle z\rangle\rangle$. We show that the generating functions of these five sequences of differential operators form an $\mathcal{N}$CS system Ω_{F_t} over $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ (see Theorem 3.15). Consequently, by the universal properties of the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$, we obtain a *differential operator specialization* $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, which can be shown is also a homomorphism of K-Hopf algebras. Then, we prove the following properties of the above differential operator specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$). First, we show in Proposition 4.1 that, for any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, the specialization $\mathcal{S}_{F_t}$ is a homomorphism of graded K-Hopf algebras from $\mathcal{N}Sym$ to the subalgebra $\mathcal{D}^{[\alpha]}\langle z\rangle \subset \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ if and only if $F_t(z) = t^{-1}F(tz)$ for some automorphism $F(z)$ of $K\langle\langle z\rangle\rangle$. Consequently, for any F_t satisfying the condition above, by taking the graded duals, we get a graded K-Hopf algebra homomorphism $\mathcal{S}_{F_t}^* : \mathcal{D}^{[\alpha]}\langle z\rangle^* \to \mathcal{Q}Sym$ from the graded dual $\mathcal{D}^{[\alpha]}\langle z\rangle^*$ of $\mathcal{D}^{[\alpha]}\langle z\rangle$ to the Hopf algebra $\mathcal{Q}Sym$ [11,17,22] of quasi-symmetric functions (see Corollary 4.2). Secondly, we show in Theorem 4.3 that, with a properly defined group product for the set $\mathbf{Hopf}_K(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ of all K-Hopf algebra homomorphisms from $\mathcal{N}Sym$, the correspondence $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ to $\mathcal{S}_{F_t} \in \mathbf{Hopf}_K(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ gives an isomorphism of groups. Finally, in Theorem 4.5, we show that the family of the specializations $\mathcal{S}_{F_t}$ with all $n \geqslant 1$ (note that n is the number of free variables z_i) and all $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ can distinguish any two NCSFs.

The arrangement of the paper is as follows. We first in Section 2 recall the definitions of the $\mathcal{N}$CS systems and the universal $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$ from NCSFs. In Section 3.1, we first fix some notations and prove some lemmas on the differential operators in commutative or noncommutative variables. In Section 3.2, for each fixed $\alpha \geqslant 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we introduce five sequences of differential operators and show that their generating functions actually form an $\mathcal{N}$CS system Ω_{F_t} over the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Consequently, we get a differential operator $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, which is also a K-Hopf algebra. In Section 4, we prove the properties of the differential operator specialization $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ that have been explained in the previous paragraph.

Finally, some remarks are as follows. This paper is the second of a sequence of papers on $\mathcal{N}$CS systems over differential operator algebras in commutative or noncommutative variables and the Grossman–Larson Hopf algebra of labeled rooted trees as well as their applications to NCSF specializations and the inversion problem. In the forthcoming paper [31], for any nonempty $W \subseteq \mathbb{N}^+$, an $\mathcal{N}$CS system $\Omega_{\mathbb{T}}^W$ over the Grossman–Larson Hopf algebra $\mathcal{H}_{GL}^W$ [9,12] of the W-labeled rooted trees will be constructed. The relations of the $\mathcal{N}$CS system $(\mathcal{H}_{GL}^W, \Omega_{\mathbb{T}}^W)$ with the $\mathcal{N}$CS systems $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$ constructed in this paper will be studied in the forthcoming paper [32]. In particular, Theorem 4.5 derived in this paper will be improved to the much smaller family of specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle z\rangle$ with all $n \geqslant 1$ and $F_t = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $H_t(z)$ is homogeneous and the Jacobian matrix JH_t is strictly lower triangular. But the proof there is based on Theorem 4.5 itself and some connections derived in [32] among the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$, $(\mathcal{H}_{GL}^W, \Omega_{\mathbb{T}}^W)$ and $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$. Finally, by the gadget mentioned in the second paragraph of this Introduction, by applying the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ to the identities of the NCSFs in the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$, we obtain a host of identities for the differential operators in the $\mathcal{N}$CS system $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$. Some of these identities and their consequences to the inversion problem [2,8] will be studied in the forthcoming paper [30]. Some other consequences of the $\mathcal{N}$CS systems $(\mathcal{H}_{GL}^W, \Omega_{\mathbb{T}}^W)$ and $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle, \Omega_{F_t})$ to the inversion problem will also be derived in [32].

2. The universal $\mathcal{N}$CS system from noncommutative symmetric functions

In this section, we first recall the definition of the $\mathcal{N}$CS systems [29] over associative algebras and some of the NCSFs (noncommutative symmetric functions) first introduced and studied in the seminal paper [10]. We then discuss the universal property of the $\mathcal{N}$CS system formed by the generating functions of these NCSFs. The main result that we will need later is Theorem 2.5 which was proved in [29].

Let K be any unital commutative $\mathbb{Q}$-algebra and A any unital associative but not necessarily commutative K-algebra. Let t be a formal central parameter, i.e. it commutes with all elements of A, and $A[\![t]\!]$ the K-algebra of formal power series in t with coefficients in A. First let us recall the following notion formulated in [29].

Definition 2.1. For any unital associative K-algebra A, a 5-tuple $\Omega = (f(t), g(t), d(t), h(t), m(t)) \in A[\![t]\!]^{\times 5}$ is said to be an $\mathcal{N}$*CS* (*noncommutative symmetric*) *system* over A if the following equations are satisfied:

$$f(0) = 1, \tag{2.1}$$

$$f(-t)g(t) = g(t)f(-t) = 1, \tag{2.2}$$

$$e^{d(t)} = g(t), \tag{2.3}$$

$$\frac{dg(t)}{dt} = g(t)h(t), \tag{2.4}$$

$$\frac{dg(t)}{dt} = m(t)g(t). \tag{2.5}$$

When the base algebra K is clear in the context, we also call the ordered pair (A, Ω) an $\mathcal{N}$*CS system*. Since $\mathcal{N}$CS systems often come from generating functions of certain elements of A that are under the consideration, the components of Ω will also be refereed as the *generating functions* of their coefficients.

All K-algebras A that we are going to work on in this paper are K-Hopf algebras. We will freely use some standard results from the theory of bialgebras and Hopf algebras, whose proofs can be found in the standard text books [1,13,19].

The following result proved in [29] later will be useful to us.

Proposition 2.2. *Let (A, Ω) be an $\mathcal{N}CS$ system as above. Suppose A is further a K-bialgebra. Then the following statements are equivalent.*

(a) *The coefficients of $f(t)$ form a sequence of divided powers of A.*
(b) *The coefficients of $g(t)$ form a sequence of divided powers of A.*
(c) *One (hence also all) of $d(t)$, $h(t)$ and $m(t)$ has all its coefficients primitive in A.*

Next, let us recall some of the NCSFs first introduced and studied in [10].

Let $\Lambda = \{\Lambda_m \mid m \geqslant 1\}$ be a sequence of noncommutative free variables and $\mathcal{N}Sym$ or $K\langle\Lambda\rangle$ the free associative algebra generated by Λ over K. For convenience, we also set $\Lambda_0 = 1$. We denote by $\lambda(t)$ the generating function of Λ_m $(m \geqslant 0)$, i.e. we set

$$\lambda(t) := \sum_{m\geqslant 0} t^m \Lambda_m = 1 + \sum_{k\geqslant 1} t^m \Lambda_m. \tag{2.6}$$

In the theory of NCSFs [10], Λ_m $(m \geqslant 0)$ is the noncommutative analog of the mth classical (commutative) elementary symmetric function and is called the *mth (noncommutative) elementary symmetric function.*

To define some other NCSFs, we consider Eqs. (2.2)–(2.5) over the free K-algebra $\mathcal{N}Sym$ with $f(t) = \lambda(t)$. The solutions for $g(t)$, $d(t)$, $h(t)$, $m(t)$ exist and are unique, whose coefficients will be the NCSFs that we are going to define. Following the notation in [10,29], we denote the resulted 5-tuple by

$$\Pi := \big(\lambda(t), \sigma(t), \Phi(t), \psi(t), \xi(t)\big) \tag{2.7}$$

and write the last four generating functions of Π explicitly as follows:

$$\sigma(t) = \sum_{m\geqslant 0} t^m S_m, \tag{2.8}$$

$$\Phi(t) = \sum_{m\geqslant 1} t^m \frac{\Phi_m}{m}, \tag{2.9}$$

$$\psi(t) = \sum_{m\geqslant 1} t^{m-1} \Psi_m, \tag{2.10}$$

$$\xi(t) = \sum_{m\geqslant 1} t^{m-1} \Xi_m. \tag{2.11}$$

Note that, by Definition 2.1, the 5-tuple Π defined above is the unique $\mathcal{N}CS$ system with $f(t) = \lambda(t)$ in Eq. (2.8) over the free K-algebra $\mathcal{N}Sym$.

Following [10], we call S_m $(m \geqslant 1)$ the *mth (noncommutative) complete homogeneous symmetric function* and Φ_m (respectively Ψ_m) the *mth power sum symmetric function of the second* (*respectively first*) *kind*. Following [29], we call $\Xi_m \in \mathcal{N}Sym$ ($m \geqslant 1$) the *mth* (*noncommutative*) *power sum symmetric function of the third kind*.

The following two propositions proved in [10,16] will be very useful for our later arguments.

Proposition 2.3. *For any unital commutative $\mathbb{Q}$-algebra K, the free algebra $\mathcal{N}Sym$ is freely generated by any one of the families of the NCSFs defined above.*

Proposition 2.4. *Let ω_Λ be the anti-involution of $\mathcal{N}Sym$ which fixes Λ_m ($m \geqslant 1$). Then, for any $m \geqslant 1$, we have*

$$\omega_\Lambda(S_m) = S_m, \tag{2.12}$$

$$\omega_\Lambda(\Phi_m) = \Phi_m, \tag{2.13}$$

$$\omega_\Lambda(\Psi_m) = \Xi_m. \tag{2.14}$$

Next, let us recall the following graded K-Hopf algebra structure of $\mathcal{N}Sym$. It has been shown in [10] that $\mathcal{N}Sym$ is the universal enveloping algebra of the free Lie algebra generated by Ψ_m ($m \geqslant 1$). Hence, it has a K-Hopf algebra structure as all other universal enveloping algebras of Lie algebras do. Its counit $\epsilon : \mathcal{N}Sym \to K$, coproduct Δ and antipode S are uniquely determined by

$$\epsilon(\Psi_m) = 0, \tag{2.15}$$

$$\Delta(\Psi_m) = 1 \otimes \Psi_m + \Psi_m \otimes 1, \tag{2.16}$$

$$S(\Psi_m) = -\Psi_m, \tag{2.17}$$

for any $m \geqslant 1$.

Next, we introduce the *weight* of NCSFs by setting the weight of any monomial $\Lambda_{m_1}^{i_1} \Lambda_{m_2}^{i_2} \cdots \Phi_{m_k}^{i_k}$ to be $\sum_{j=1}^{k} i_j m_j$. For any $m \geqslant 0$, we denote by $\mathcal{N}Sym_{[m]}$ the vector subspace of $\mathcal{N}Sym$ spanned by the monomials of Λ of weight m. Then it is easy to see that

$$\mathcal{N}Sym = \bigoplus_{m \geqslant 0} \mathcal{N}Sym_{[m]}, \tag{2.18}$$

which provides a grading for $\mathcal{N}Sym$.

Note that it has been shown in [10], for any $m \geqslant 1$, the NCSFs $S_m, \Phi_m, \Psi_m \in \mathcal{N}Sym_{[m]}$. By Proposition 2.4, this is also true for the NCSFs Ξ_m's. By the facts above and Eqs. (2.15)–(2.17), it is also easy to check that, with the grading given in Eq. (2.18), $\mathcal{N}Sym$ forms a graded K-Hopf algebra. Its graded dual is given by the space $\mathcal{Q}Sym$ of quasi-symmetric functions, which were first introduced by I. Gessel [11] (see [17,22] for more discussions).

Now we come back to our discussions on the $\mathcal{N}$CS systems. From the definitions of the NCSFs above, we see that $(\mathcal{N}Sym, \Pi)$ obviously forms an $\mathcal{N}$CS system. More importantly, as shown in Theorem 2.1 in [29], we have the following important theorem on the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$.

Theorem 2.5. *Let A be a K-algebra and Ω an $\mathcal{N}CS$ system over A. Then,*

(a) *There exists a unique K-algebra homomorphism $\mathcal{S}:\mathcal{N}Sym \to A$ such that $\mathcal{S}^{\times 5}(\Pi) = \Omega$.*
(b) *If A is further a K-bialgebra (respectively K-Hopf algebra) and one of the equivalent statements in Proposition 2.2 holds for the $\mathcal{N}CS$ system Ω, then $\mathcal{S}:\mathcal{N}Sym \to A$ is also a homomorphism of K-bialgebras (respectively K-Hopf algebras).*

Remark 2.6. By applying the similar arguments as in the proof of Theorem 2.5, or simply taking the quotient over the two-sided ideal generated by the commutators of Λ_m's, it is easy to see that, over the category of commutative K-algebras, the universal $\mathcal{N}CS$ system is given by the generating functions of the corresponding classical (commutative) symmetric functions [18].

Remark 2.7. One direct consequence of Theorem 2.5 above is as follows. Note that the relations or polynomial identities between any two families of NCSFs in the first four components of Π have been given explicitly in [10]. By applying the anti-automorphism ω_λ in Proposition 2.4, one can easily derive the relations of the NCSFs Ξ_m's with other NCSFs in Π (for example, see §4.1 in [30] for a complete list). By applying the homomorphism $\mathcal{S}$, we get a host of identities among the corresponding elements of A. This will be a very effective method to prove identities for the elements of A which are involved in an $\mathcal{N}CS$ system over A. On the other hand, if the $\mathcal{N}CS$ system (A, Ω) has already been well understood, the homomorphism $\mathcal{S}:\mathcal{N}Sym \to A$, usually called a *specialization* of NCSFs, can be used to study certain properties of NCSFs.

3. $\mathcal{N}CS$ systems over differential operator algebras

Let K be any unital commutative $\mathbb{Q}$-algebra and $z = (z_1, z_2, \ldots, z_n)$ free variables, i.e. commutative or noncommutative independent variables. Let t be a formal central parameter, i.e. it commutes with z and all elements of K. We denote by $K\langle\langle z\rangle\rangle$ and $K[[t]]\langle\langle z\rangle\rangle$ the K-algebras of formal power series in z over K and $K[[t]]$, respectively.[1] For any positive integer $\alpha \geqslant 1$, we let $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ denote the K-algebra of the differential operators of $K\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$. We also fix the notation $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ for the set of all the automorphisms $F_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ over $K[[t]]$ which have the form $F(z) = z - H_t(z)$ for some $H_t(z) \in K[[t]]\langle\langle z\rangle\rangle^{\times n}$ with $o(H_t(z)) \geqslant \alpha$ and $H_{t=0}(z) = 0$. It is easy to check that $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ actually forms a subgroup of the automorphism group of $K[[t]]\langle\langle z\rangle\rangle$.

In Section 3.1, we fix more notation and prove some simple results on the differential operators in commutative or noncommutative free variables z. In Section 3.2, for any fixed automorphism $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we introduce five families of differential operators associated with $F_t(z)$ and its inverse $G_t(z) := F_t^{-1}(z)$ and show that their generating functions form an $\mathcal{N}CS$ system Ω_{F_t} over the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Consequently, by the universal property (see Theorem 2.5) of the $\mathcal{N}CS$ system $(\mathcal{N}Sym, \Pi)$ from NCSFs, we obtain a family of differential operator specializations $\mathcal{S}_{F_t}:\mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ $(F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle)$ for NCSFs. More properties of the specializations $\mathcal{S}_{F_t}$ $(F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle)$ will be studied in the next section.

[1] Since most of the results as well as their proofs in this paper do not depend on the commutativity of the free variables z, we will not distinguish the commutative and the noncommutative case, unless stated otherwise, and adapt the notations for noncommutative variables uniformly for the both cases.

3.1. Differential operators in (noncommutative) free variables

Let K, z and t be as fixed above. In this subsection, we mainly fix more notations and prove some simple results on the differential operators in z. All the results proved in this section should be well known, especially in the commutative case. But for the completeness, especially when the noncommutative case is concerned, we also include proofs here. As we mentioned early, in this subsection as well as in the rest of this paper, we do not assume the commutativity of our free variables z, unless stated otherwise.

Recall that a *K-derivation* or simply a *derivation* of $K\langle\langle z\rangle\rangle$ is a K-linear map $\delta : K\langle\langle z\rangle\rangle \to K\langle\langle z\rangle\rangle$ which satisfies the Leibnitz rule, i.e. for any $f, g \in K\langle\langle z\rangle\rangle$, we have

$$\delta(fg) = (\delta f)g + f(\delta g). \tag{3.1}$$

We denote by $\mathcal{D}er_K\langle\langle z\rangle\rangle$ or $\mathcal{D}er\langle\langle z\rangle\rangle$, when the base algebra K is clear in the context, the set of all K-derivations of $K\langle\langle z\rangle\rangle$. The unital subalgebra of $\mathrm{End}_K(K\langle\langle z\rangle\rangle)$ (the set of endomorphisms of $K\langle\langle z\rangle\rangle$ as a K-vector space, not as a K-algebra) generated by all K-derivations of $K\langle\langle z\rangle\rangle$ is denoted by $\mathcal{D}_K\langle\langle z\rangle\rangle$ or $\mathcal{D}\langle\langle z\rangle\rangle$. Elements of $\mathcal{D}\langle\langle z\rangle\rangle$ are called the (*formal*) *differential operators* in the free variables z.

For any $\alpha \geqslant 1$, we denote by $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ the set of K-derivations of $K\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$. The unital subalgebra of $\mathcal{D}\langle\langle z\rangle\rangle$ generated by elements of $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ will be denoted by $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Note that, by the definitions above, the operators of scalar multiplications are in $\mathcal{D}\langle\langle z\rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, but not in $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$. When the base algebra is $K[\![t]\!]$ instead of K itself, the corresponding notation $\mathcal{D}er\langle\langle z\rangle\rangle$, $\mathcal{D}\langle\langle z\rangle\rangle$, $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ will be denoted by $\mathcal{D}er_t\langle\langle z\rangle\rangle$, $\mathcal{D}_t\langle\langle z\rangle\rangle$, $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z\rangle\rangle$, respectively. For example, $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$ stands for the set of all $K[\![t]\!]$-derivations of the $K[\![t]\!]$-algebra $K[\![t]\!]\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$. Note that $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle = \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z\rangle\rangle = \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$.

For any $1 \leqslant i \leqslant n$ and $u(z) \in K\langle\langle z\rangle\rangle$, we denote by $[u(z)\frac{\partial}{\partial z_i}]$ [2] the K-derivation which maps z_i to $u(z)$ and z_j to 0 for any $j \neq i$. For any $\vec{u} = (u_1, u_2, \ldots, u_n) \in K\langle\langle z\rangle\rangle^{\times n}$, we set

$$\left[\vec{u}\frac{\partial}{\partial z}\right] := \sum_{i=1}^{n} \left[u_i \frac{\partial}{\partial z_i}\right]. \tag{3.2}$$

Warning. *When z are noncommutative free variables, we in general do* not *have $[u(z)\frac{\partial}{\partial z_i}]g(z) = u(z)\frac{\partial g}{\partial z_i}$ for all $u(z), g(z) \in K\langle\langle z\rangle\rangle$. For example, let $g = z_j z_i$ with $j \neq i$, we have*

$$\left[u\frac{\partial}{\partial z_i}\right](z_j z_i) = \left(\left[u\frac{\partial}{\partial z_i}\right]z_j\right)z_i + z_j\left(\left[u\frac{\partial}{\partial z_i}\right]z_i\right) = z_j u(z),$$
$$u(z)\frac{\partial g}{\partial z_i} = u(z) z_j,$$

which are not equal to each other unless $u(z)$ commutes with z_j.

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[2] The reason we put a bracket $[\cdot]$ in the notation for this derivation of $K\langle\langle z\rangle\rangle$ is to avoid any possible confusion caused by a subtle point described in the *Warning* below.

With the notation above, it is easy to see that any K-derivation δ of $K\langle\langle z\rangle\rangle$ can be written uniquely as $\sum_{i=1}^n [f_i(z)\frac{\partial}{\partial z_i}]$ with $f_i(z)=\delta\cdot z_i\in K\langle\langle z\rangle\rangle$ $(1\leqslant i\leqslant n)$. Also, in both commutative and noncommutative cases, we have the following Lie bracket relation in the noncommutative case, namely, for any derivations $\delta=[f(z)\frac{\partial}{\partial z}]$ and $\eta=[g(z)\frac{\partial}{\partial z}]$ with $f,g\in K\langle\langle z\rangle\rangle$, we have

$$[\delta,\eta]=\left[(\delta g)\frac{\partial}{\partial z}\right]-\left[(\eta f)\frac{\partial}{\partial z}\right], \tag{3.3}$$

where $[\delta,\eta]$ is the commutator of δ and η.

With the bracket above, $\mathcal{D}er\langle\langle z\rangle\rangle$ forms a Lie algebra and its universal enveloping algebra is exactly the differential operator algebra $\mathcal{D}\langle\langle z\rangle\rangle$. Consequently, $\mathcal{D}\langle\langle z\rangle\rangle$ has a K-Hopf algebra structure as all other enveloping algebras of Lie algebras do. In particular, its coproduct Δ, antipode S and counit ϵ are respectively determined by the following properties: for any $\delta\in\mathcal{D}er\langle\langle z\rangle\rangle$,

$$\Delta(\delta)=1\otimes\delta+\delta\otimes 1, \tag{3.4}$$

$$S(\delta)=-\delta, \tag{3.5}$$

$$\epsilon(\delta)=\delta\cdot 1. \tag{3.6}$$

Next, we introduce the following two operations for the K-derivations of $K\langle\langle z\rangle\rangle$.

First, for any $\phi,\delta\in\mathcal{D}er\langle\langle z\rangle\rangle$ with $\delta=[f(z)\frac{\partial}{\partial z}]$ for some $f(z)\in K\langle\langle z\rangle\rangle$, we set

$$\phi\rhd\delta:=\left[(\phi f)(z)\frac{\partial}{\partial z}\right]. \tag{3.7}$$

Secondly, for any K-derivations $\delta_i=[\vec{v}_i(z)\frac{\partial}{\partial z}]$ $(1\leqslant i\leqslant m)$ with $\vec{v}_i(z)\in K\langle\langle z\rangle\rangle^{\times n}$, we define a new linear operator $B_+(\delta_1,\delta_2,\ldots,\delta_m)$ as follows. Let $w=(w_1,w_2,\ldots,w_n)$ be another n free variables which are independent and do not commute with the free variables z. We define $B_+(\delta_1,\delta_2,\ldots,\delta_m)$ by setting, for any $u(z)\in K\langle\langle z\rangle\rangle$,

$$\begin{aligned}&B_+(\delta_1,\delta_2,\ldots,\delta_m)u(z)\\&\quad:=\left[\vec{v}_1(w)\frac{\partial}{\partial z}\right]\left[\vec{v}_2(w)\frac{\partial}{\partial z}\right]\cdots\left[\vec{v}_m(w)\frac{\partial}{\partial z}\right]u(z)\bigg|_{w=z}.\end{aligned} \tag{3.8}$$

Furthermore, for any $k_i\geqslant 0$ $(1\leqslant i\leqslant m)$, we let $B_+(\delta_1^{k_1},\delta_2^{k_2},\ldots,\delta_m^{k_m})$ denote the operator obtained by applying B_+ to the multi-set of j_1-copies of δ_1, j_2-copies of $\delta_2,\ldots,j_m$-copies of δ_m.

Note that $B_+(\delta_1,\delta_2,\ldots,\delta_m)$ is multi-linear and symmetric in the components δ_i $(1\leqslant i\leqslant m)$. When $m=1$, we have $B_+(\delta_1)=\delta_1$.

Next, we show that $B_+(\delta_1,\delta_2,\ldots,\delta_m)$ is still a differential operator in z, i.e. $B_+(\delta_1,\delta_2,\ldots,\delta_m)\in\mathcal{D}\langle\langle z\rangle\rangle$. But first we need prove the following lemma.

Lemma 3.1. *For any $\phi, \delta_i \in \mathcal{D}er\langle\langle z\rangle\rangle$ $(1 \leqslant i \leqslant m)$, we have*

$$\phi \cdot B_+(\delta_1, \delta_2, \ldots, \delta_m) = B_+(\phi, \delta_1, \delta_2, \ldots, \delta_m) + \sum_{i=1}^{m} B_+(\delta_1, \ldots, \phi \rhd \delta_i, \ldots, \delta_m). \tag{3.9}$$

Proof. First, since $B_+(\delta_1, \delta_2, \ldots, \delta_m)$ is multi-linear in the components δ_i's, we may assume $\delta_i = [a_i(z)\frac{\partial}{\partial z_{k_i}}]$ $(1 \leqslant i \leqslant m)$ for some $a_i(z) \in K\langle\langle z\rangle\rangle$ and $1 \leqslant k_i \leqslant n$. Furthermore, to show Eq. (3.9), we only need to show its two sides have the same values at all monomials of z.

Now, set $\Psi := B_+(\delta_1, \delta_2, \ldots, \delta_m)$ and let $u(z)$ be any monomial of z. By Eq. (3.8), we know that $\Psi u(z)$ is the sum of all the terms obtained by replacing m-copies of z_{k_i}'s in the monomial $u(z)$ by the corresponding $a_i(z)$'s in all possible ways. Consequently, each of these terms is a monomial in z_i's and m-copies of $a_i(z)$'s. Now, we apply the derivation ϕ to $\Psi u(z)$. By the Leibnitz rule, we know that ϕ either lands on a variable z_i or a copy of $a_i(z)$'s. It is easy to check that the sum of all the terms obtained in the former case is the same as $B_+(\phi, \delta_1, \delta_2, \ldots, \delta_m)u(z)$; while, the sum of all the terms obtained in the later case is the same as those obtained by applying $\sum_{i=1}^{m} B_+(\delta_1, \ldots, \phi \rhd \delta_i, \ldots, \delta_m)$ to $u(z)$. □

Corollary 3.2. *For any $\alpha \geqslant 1$ and $\delta_i \in \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ $(1 \leqslant i \leqslant m)$, we have $B_+(\delta_1, \delta_2, \ldots, \delta_m) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.*

Proof. We use the mathematical induction on $m \geqslant 1$. When $m = 1$, we have $B_+(\delta_1) = \delta_1$, hence nothing needs to prove.

Now, let $m \geqslant 2$. By Eq. (3.9), we have

$$B_+(\delta_1, \ldots, \delta_m) = \delta_1 B_+(\delta_2, \ldots, \delta_m) - \sum_{i=2}^{m} B_+(\delta_2, \ldots, \delta_1 \rhd \delta_i, \ldots, \delta_m).$$

Note that, by Eq. (3.7), $\delta_1 \rhd \delta_i$ $(2 \leqslant i \leqslant m)$ is still a derivation in $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$. Therefore, from the equation above and the induction assumption, we see that $B_+(\delta_1, \delta_2, \ldots, \delta_m) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. □

Next, let us consider the Taylor series expansions for formal power series in commutative or noncommutative variables z. First, we have the following lemma which can be proved by a similar argument as in the commutative case.

Lemma 3.3. *Let $v = (v_1, v_2, \ldots, v_n)$ be another n free variables which are independent and do not commute with z. Let t be a formal parameter which commutes with both z and v. Then, for any $u(z) \in K\langle\langle z\rangle\rangle$, we have*

$$u(z + tv) = \sum_{k \geqslant 0} \frac{t^k}{k!} B_+\left(\left[v\frac{\partial}{\partial z}\right]\right)^k u(z). \tag{3.10}$$

Proposition 3.4. *For any $\alpha \geqslant 1$ and $F_t(z) = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we have the following Taylor series expansion for any $u(z) \in K\langle\langle z\rangle\rangle$:*

$$u\big(F_t(z)\big) = \sum_{k \geqslant 0} \frac{(-1)^k}{k!} B_+\left(\left[H_t(z)\frac{\partial}{\partial z}\right]^k\right) u(z). \tag{3.11}$$

Proof. The proposition follows directly by setting $tv = -H_t(z)$ in Eq. (3.10) and then using the definition of the operation B_+ in Eq. (3.8). □

Finally, let us prove the following lemma which will be needed later.

Lemma 3.5. *Let $z = (z_1, z_2, \dots, z_n)$ be commutative free variables and $\Phi \in \mathcal{D}\langle\langle z\rangle\rangle$. Suppose that, there exists $N > 0$ such that $\Phi u(z) = 0$ for any $u(z) \in K\langle\langle z\rangle\rangle$ with $o(u(z)) \geqslant N$. Then $\Phi = 0$.*

Proof. First, let $\mathrm{End}_K(K\langle\langle z\rangle\rangle)$ be the set of K-linear maps from $K\langle\langle z\rangle\rangle$ to $K\langle\langle z\rangle\rangle$ and $\mathcal{A}$ the unital subalgebra of $\mathrm{End}_K(K\langle\langle z\rangle\rangle)$ generated by K-derivations and the linear operators given by multiplications by elements of $K\langle\langle z\rangle\rangle$. Let $\mathcal{N}$ be the set of all elements $\Psi \in \mathcal{A}$ such that, for some $N \geqslant 1$ (depending on Ψ), $\Psi u(z) = 0$ for any $u(z) \in K\langle\langle z\rangle\rangle$ with $o(u(z)) \geqslant N$. It is straightforward to check that $\mathcal{N}$ forms a left ideal of $\mathcal{A}$. Furthermore, for any polynomial $b(z) \in K\langle\langle z\rangle\rangle$, we denote by $L_{b(z)}$ the operator of multiplication by $b(z)$. Then, it is easy to check that we also have $\mathcal{N}L_{b(z)} \subset \mathcal{N}$.

Next we use induction on the *order* $\mathrm{Ord}(\Phi)$ of Φ to show that $\mathcal{N} = 0$. First, if $\mathrm{Ord}(\Phi) = 0$, then Φ is just an operator of multiplication by an element of $K\langle\langle z\rangle\rangle$, and the lemma obviously holds. Assume the lemma holds for all $\Phi \in \mathcal{N}$ with $\mathrm{Ord}(\Phi) \leqslant m$. Consider the case that $\mathrm{Ord}(\Phi) = m + 1$. Let $\delta := (\delta_1, \delta_2, \dots, \delta_k)$ be a sequence of K-derivations such that Φ can be written as

$$\Phi = \sum_{\substack{I\in\mathbb{N}^k\\ |I|\leqslant m+1}} a_I(z)\delta^I \tag{3.12}$$

for some $a_I(z) \in K\langle\langle z\rangle\rangle$.

Note that, in general, we have

$$\Phi L_{z_1} = [\Phi, L_{z_1}] + L_{z_1}\Phi, \tag{3.13}$$

where $[\Phi, L_{z_1}]$ denotes the commutator of the operators Φ and L_{z_1}.

First, by using the form of Φ in Eq. (3.12), it is easy to check that $\mathrm{Ord}([\Phi, L_{z_1}]) \leqslant \mathrm{Ord}(\Phi) - 1$. Secondly, by Eq. (3.13) and the facts about $\mathcal{N}$ mentioned in the first paragraph of the proof, we have $\Phi L_{z_1} \in \mathcal{N}$ and then $[\Phi, L_{z_1}] \in \mathcal{N}$. By our induction assumption, we have $[\Phi, L_{z_1}] = 0$. Therefore, Φ commutes with the left multiplication by z_1. Hence it also commutes with the left multiplication by z_1^m for any $m \geqslant 1$. Now let N be a positive integer such that $\Phi \cdot u(z) = 0$ for any $u(z) \in K\langle\langle z\rangle\rangle$ with $o(u(z)) \geqslant N$. Then, for any $f(z) \in K\langle\langle z\rangle\rangle$, we have $\Phi(z_1^N f(z)) = z_1^N \Phi f(z) = 0$. Therefore, $\Phi f(z) = 0$ for any $f(z) \in K\langle\langle z\rangle\rangle$. Hence $\Phi = 0$. □

3.2. $\mathcal{N}$CS systems over differential operator algebras

Let K, z, t, $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($\alpha \geqslant 1$) be as fixed in the previous subsection. Recall that we also have defined $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ to be the set of all the automorphisms $F_t(z)$ of $K[\![t]\!]\langle\langle z\rangle\rangle$ over $K[\![t]\!]$, which are of the form

$$F_t(z) = z - H_t(z) \tag{3.14}$$

for some $H_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle^{\times n}$ with $o(H_t(z)) \geqslant \alpha$ and $H_{t=0}(z) = 0$. Note that, for any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ as above, its inverse map can always be written uniquely as

$$G_t(z) := F_t^{-1}(z) = z + M_t(z) \tag{3.15}$$

for some $M_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle^{\times n}$ with $o(M_t(z)) \geqslant \alpha$ and $M_{t=0}(z) = 0$. Throughout the rest of this section, we will fix an arbitrary $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ and always let $H_t(z)$, $G_t(z)$ and $M_t(z)$ be determined as in Eqs. (3.14) and (3.15).

Note that $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ can be viewed as a deformation parameterized by t of the formal map $F(z) := F_{t=1}(z)$, when it makes sense. For more studies of $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ from the deformation point of view, see [27] and [28]. Actually, the construction of the $\mathcal{N}$CS system given in this subsection is mainly motivated by and also depends on the studies of $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ given in [27,28].

First, let us introduce the following five sequences of differential operators associated with the fixed automorphism $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ and its inverse $G_t(z)$. As we will see later, the generating functions of these differential operators will form an $\mathcal{N}$CS system over the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

The first two sequences of differential operators come from $F_t(z)$ and $G_t(z)$ as follows.

Lemma 3.6. *There exist unique sequences $\{\lambda_m \mid m \geqslant 0\}$ and $\{s_m \mid m \geqslant 0\}$ of elements of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ with $\lambda_0 = s_0 = 1$ such that, for any $u_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$, we have*

$$\left(\sum_{m=0}^{\infty}(-1)^m t^m \lambda_m\right) u_t(z) = u_t(F_t), \tag{3.16}$$

$$\left(\sum_{m=0}^{\infty} t^m s_m\right) u_t(z) = u_t(G_t). \tag{3.17}$$

The signs appearing in Eq. (3.16) as well as somewhere else in this subsection are chosen in such a way that later in Theorem 3.15 the correspondence $\mathcal{S}_{F_t}$ between the NCSFs in the universal $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$ and the differential operators defined in this subsection will be in the simplest form.

Proof. First, let us show the uniqueness. By Eqs. (3.16) and (3.17), for any $m \geqslant 0$ and $u(z) \in K\langle\langle z\rangle\rangle \subset K[\![t]\!]\langle\langle z\rangle\rangle$, we have

$$\lambda_m u(z) = \operatorname{Res}_{t=0} u(F_t) t^{-m-1},$$

$$s_m u(z) = \operatorname{Res}_{t=0} u(G_t) t^{-m-1}.$$

Hence, as differential operators of $K\langle\langle z\rangle\rangle$, λ_m and s_m $(m \geqslant 1)$ are uniquely determined by Eqs. (3.16) and (3.17), respectively.

To show the existence, we first consider Eqs. (3.16) and (3.17) with $u_t(z) \in K\langle\langle z\rangle\rangle$, i.e. $u_t(z)$ does not depend on t. By Corollary 3.4, for any $u(z) \in K\langle\langle z\rangle\rangle$, we have

$$u\big(F_t(z)\big) = \left(\sum_{k\geqslant 0} \frac{(-1)^k}{k!} B_+\left(\left[H_t(z)\frac{\partial}{\partial z}\right]^k\right)\right) u(z). \tag{3.18}$$

Note that, by Corollary 3.2 and the condition $o(H_t(z)) \geqslant \alpha$ (since $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$), all the operators involved in the equation above are differential operators in $\mathcal{D}_t^{[\alpha]}\langle\langle z\rangle\rangle$ which is the same as $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$. Therefore, the bracketed sum of the differential operators in Eq. (3.18) above can be written as $\sum_{m=0}^{\infty}(-1)^m t^m \lambda_m$ for some $\lambda_m \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ $(m \geqslant 0)$. With the new differential operators λ_m's, Eq. (3.18) becomes

$$u\big(F_t(z)\big) = \left(\sum_{m=0}^{\infty}(-1)^m t^m \lambda_m\right) u(z). \tag{3.19}$$

Note that $H_{t=0}(z) = 0$ since $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. By setting $t = 0$ in Eq. (3.19) above, we get $\lambda_0 = 1$.

Now we show Eq. (3.19) also holds for any $u_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$. First, we write $u_t(z) = \sum_{k\geqslant 0} u_k(z)t^k$ with $u_k(z) \in K\langle\langle z\rangle\rangle$ $(k \geqslant 0)$ and then apply Eq. (3.19) to each $u_k(z) \in K\langle\langle z\rangle\rangle$. Then, it is easy to see that Eq. (3.19) also holds for $u_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$. Therefore, we have proved the existence of the sequence $\{\lambda_m \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle \mid m \geqslant 0\}$. The existence of the sequence $\{s_m \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle \mid m \geqslant 0\}$ can be proved similarly with F_t replaced by G_t. $\square$

We denote by $f(t)$ and $g(t)$ the generating functions of the differential operators λ_m and s_m $(m \geqslant 0)$, respectively, i.e. we set

$$f(t) := \sum_{m=0}^{\infty} t^m \lambda_m, \tag{3.20}$$

$$g(t) := \sum_{m=0}^{\infty} t^m s_m. \tag{3.21}$$

We will also view $f(t)$ and $g(t)$ as differential operators of $K[\![t]\!]\langle\langle z\rangle\rangle$ in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Furthermore, for any $u_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$, by Eqs. (3.16) and (3.17), we have

$$f(-t)u_t(z) = u_t(F_t), \tag{3.22}$$

$$g(t)u_t(z) = u_t(G_t). \tag{3.23}$$

From the proof of Lemma 3.6, we see $f(-t)$ and $g(t)$ can be given as follows.

Corollary 3.7.

$$f(-t) = \sum_{k\geqslant 0} \frac{(-1)^k}{k!} B_+\left(\left[H_t(z)\frac{\partial}{\partial z}\right]^k\right), \tag{3.24}$$

$$g(t) = \sum_{k\geqslant 0} \frac{1}{k!} B_+\left(\left[M_t(z)\frac{\partial}{\partial z}\right]^k\right). \tag{3.25}$$

Lemma 3.8.

$$g(t)f(-t) = f(-t)g(t) = 1. \tag{3.26}$$

Proof. For any $u(z) \in K\langle\langle z\rangle\rangle$, by Eqs. (3.22), (3.23) and the fact that $G_t(z) = F_t^{-1}(z)$, we have

$$g(t)f(-t)u(z) = g(t)u\big(F_t(z)\big) = u\big(F_t\big(G_t(z)\big)\big) = u(z).$$

Hence, as elements of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$, we have $g(t)f(-t) = \mathrm{id} = 1$. Similarly, we can show $f(-t)g(t) = 1$. □

Next, let us consider the *D-Log* of $F_t(z)$, which has been studied in [5–7,20,25] and [26] for the commutative case.

Lemma 3.9. *For any $\alpha \geqslant 1$, $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, there exists a unique $a_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle^{\times n}$ with $a_{t=0}(z) = 0$ and $o(a_t(z)) \geqslant \alpha$ such that*

$$e^{[a_t(z)\frac{\partial}{\partial z}]} \cdot z = F_t(z), \tag{3.27}$$

where, as usual, the exponential in the equation above is given by

$$e^{[a_t(z)\frac{\partial}{\partial z}]} = \sum_{m\geqslant 0} \frac{1}{m!}\left[a_t(z)\frac{\partial}{\partial z}\right]^m. \tag{3.28}$$

For the proof of the lemma, we refer the reader to the proof of Proposition 2.1 in [26] which gives an elementary proof of this result for the commutative case. The main idea of the proof is to solve the homogeneous (in z) parts of $a_t(z)$ recursively from Eq. (3.27). Even though the proof in [26] is for the commutative case over $\mathbb{C}$, it works equally well for the noncommutative case as long as the base algebra, which is $K[\![t]\!]$ in our case, is a $\mathbb{Q}$-algebra.

Following the terminology used in [5–7,25] for the commutative case, we call $a_t(z)$ the *D-Log* of $F_t(z)$. Now, from the D-Log $a_t(z)$, we define a sequence $\{\phi_m \in \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle \mid m \geqslant 1\}$ of the derivations of $K\langle\langle z\rangle\rangle$ by requiring

$$d(t) := \sum_{m=1}^{\infty} \frac{t^m}{m}\phi_m = -\left[a_t(z)\frac{\partial}{\partial z}\right]. \tag{3.29}$$

Lemma 3.10.

$$e^{d(t)} = g(t). \tag{3.30}$$

Proof. First, note that, it is well known that the exponential of any derivation of an algebra, when it is well defined, is an automorphism of the algebra. By this fact and Eqs. (3.27) and (3.22), we have, for any polynomial $u(z)$ in the free variables z,

$$\begin{aligned} e^{[a_t(z)\frac{\partial}{\partial z}]}u(z) &= u\big(e^{[a_t(z)\frac{\partial}{\partial z}]}z\big) \\ &= u\big(F_t(z)\big) \\ &= f(-t)u(z). \end{aligned}$$

Hence, as elements of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$, we must have

$$e^{[a_t(z)\frac{\partial}{\partial z}]} = f(-t). \tag{3.31}$$

Secondly, by Eq. (3.29), we have $e^{[a_t(z)\frac{\partial}{\partial z}]} = e^{-d(t)}$ whose inverse element in $\mathcal{D}\langle\langle z\rangle\rangle[\![t]\!]$ is obviously $e^{d(t)}$; while by Eq. (3.26), the inverse element of $f(-t)$ is given by $g(t)$. Hence, by Eq. (3.31) above, we have $e^{d(t)} = g(t)$. □

One remark on the D-Log is as follows. Note that, by taking the D-Log, we have a well-defined map

$$\begin{aligned} T: \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle &\to t\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle, \\ F_t(z) &\to \left[a_t(z)\frac{\partial}{\partial z}\right]. \end{aligned} \tag{3.32}$$

Conversely, for any $\delta_t \in t\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle = t\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$, we can always write δ_t uniquely as $\delta_t = [a_t(z)\frac{\partial}{\partial z}]$ for some $a_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$ with $a_{t=0}(z) = 0$ and $o(a_t(z)) \geqslant \alpha$. By taking the exponential $e^{\delta_t} = e^{[a_t(z)\frac{\partial}{\partial z}]}$ of the derivation δ_t, we get an automorphism of $K[\![t]\!]\langle\langle z\rangle\rangle$. From Eq. (3.28), it is easy to see that the resulted automorphism $e^{[a_t(z)\frac{\partial}{\partial z}]}$ actually lies in $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Therefore we have the following proposition.

Proposition 3.11. *For any $\alpha \geqslant 1$, $T: \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle \to t\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$ defined in Eq.* (3.32) *is a bijection, whose inverse map is given by the exponential map.*

By using the Baker–Campbell–Hausdorff formula (see [24] or [21]), it is easy to check that, formally, the Lie algebra of the Lie group $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ is exactly the Lie algebra $t\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$. The inverse map of T, which is the exponential map given by Eq. (3.28), is the same as the exponential map of the Lie group $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ from its Lie algebra to itself. So, in terms of the language of Lie theory, the proposition above just says that the exponential map of the formal Lie group $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ turns out to be a bijection whose inverse map is given by taking the D-Log of the elements of $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$.

Next, we define the last two sequences $\{\psi_m \mid m \geqslant 1\}$ and $\{\xi_m \mid m \geqslant 1\}$ of elements of $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ by requiring

$$h(t) := \sum_{m\geqslant 1}\psi_m t^{m-1} = \left[\frac{\partial M_t}{\partial t}(F_t)\frac{\partial}{\partial z}\right], \tag{3.33}$$

$$m(t) := \sum_{m\geqslant 1}\xi_m t^{m-1} = \left[\frac{\partial H_t}{\partial t}(G_t)\frac{\partial}{\partial z}\right]. \tag{3.34}$$

With the facts that $o(H_t(z)) \geqslant \alpha$ and $o(M_t(z)) \geqslant \alpha$, it is easy to see that the derivations ψ_m and ξ_m $(m \geqslant 1)$ defined above are indeed in $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$.

To get some concrete ideas for the differential operators defined in Eqs. (3.33) and (3.34), let us recall the following result proved in Lemma 4.1 in [28] for the special automorphism $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ of the form $F_t(z) = z - tH(z)$ with $H(z)$ independent on t, i.e. $H(z) \in K\langle\langle z\rangle\rangle^{\times n}$.

Lemma 3.12. *For any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ of the form $F_t(z) = z - tH(z)$ as above, let $N_t(z) = t^{-1}M_t(z)$. Then we have*

$$m(t) = \left[N_t(z)\frac{\partial}{\partial z}\right], \tag{3.35}$$

$$h(t) = \sum_{m \geqslant 1} t^{m-1}\left[C_m(z)\frac{\partial}{\partial z}\right], \tag{3.36}$$

where $C_m(z) \in K\langle\langle z\rangle\rangle^{\times n}$ $(m \geqslant 1)$ are defined recurrently by

$$C_1(z) = H(z), \tag{3.37}$$

$$C_m(z) = \left[C_{m-1}(z)\frac{\partial}{\partial z}\right]H(z), \tag{3.38}$$

for any $m \geqslant 2$.

Consequently, for any $m \geqslant 1$, the derivations ψ_m and ξ_m defined in Eqs. (3.34) *and* (3.33) *are given by*

$$\psi_m = \left[C_m(z)\frac{\partial}{\partial z}\right], \tag{3.39}$$

$$\xi_m = \left[N_{[m]}(z)\frac{\partial}{\partial z}\right], \tag{3.40}$$

where $N_{[m]}(z) \in K\langle\langle z\rangle\rangle^{\times n}$ $(m \geqslant 1)$ is the coefficient of t^{m-1} of $N_t(z)$.

By the mathematical induction on $m \geqslant 1$, it is easy to show that, when z are commutative variables, we further have

$$C_m(z) = (JH)^{m-1}H(z) \tag{3.41}$$

for any $m \geqslant 1$, where JH is the Jacobian matrix of $H(z) \in K[[z]]^{\times n}$.

One remark about Lemma 3.12 is as follows. For the automorphisms $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ of the form $F_t(z) = z - tH(z)$ as above, the differential operators ψ_m $(m \geqslant 1)$ have the simplest form. In particular, in the commutative case, by Eq. (3.41), they capture the nilpotence of the Jacobian matrix JH in a nice way, namely, JH is nilpotent iff $\psi_m = 0$ for any $m \geqslant n$. On the other hand, the differential operators ξ_m $(m \geqslant 1)$ capture the inverse map $G_t(z)$ of $F_t(z)$ directly. Together with the main result of this section (see Theorem 3.15 below), this leads to an interesting connection of the NCSF theory with the inversion problem [2,8], i.e. the problem that studies various properties of the inverse map G_t from F_t. This connection will be discussed more in Remark 3.16 at the end of this subsection.

Now let us consider the relations of the differential operators $h(t)$ and $m(t)$ defined in Eqs. (3.33) and (3.34), respectively, with the differential operator $g(t)$ defined by Eq. (3.21) or (3.17).

Lemma 3.13.

$$\frac{dg(t)}{dt} = g(t)h(t), \tag{3.42}$$

$$\frac{dg(t)}{dt} = m(t)g(t). \tag{3.43}$$

Proof. First, by Proposition 3.3 in [28], for any $u(z) \in K\langle\langle z\rangle\rangle$, the following equations hold:

$$\frac{\partial u(G_t)}{\partial t} = \big(h(t)u\big)(G_t), \tag{3.44}$$

$$\frac{\partial u(G_t)}{\partial t} = m(t)u(G_t). \tag{3.45}$$

Now, by Eq. (3.23), we can rewrite Eq. (3.44) as follows:

$$\frac{\partial}{\partial t}\big(g(t)u(z)\big) = g(t)\big(h(t)u\big)(z),$$

$$\frac{dg(t)}{dt}u(z) = \big(g(t)h(t)\big)u(z). \tag{3.46}$$

Since Eq. (3.46) above holds for any $u(z) \in K\langle\langle z\rangle\rangle$, $\frac{dg(t)}{dt}$ and $g(t)h(t)$, as elements of $\mathcal{D}\langle\langle z\rangle\rangle[[t]]$, must be the same. Hence we have Eq. (3.42).

To show Eq. (3.43), again by Eq. (3.23), we can rewrite Eq. (3.45) as follows:

$$\frac{\partial}{\partial t}\big(g(t)u(z)\big) = m(t)\big(g(t)u\big)(z),$$

$$\frac{dg(t)}{dt}u(z) = \big(m(t)g(t)\big)u(z).$$

By the same reason as above, $\frac{dg(t)}{dt}$ and $m(t)g(t)$ must be the same. Hence we get Eq. (3.43). $\square$

Now we can formulate the main result of this subsection. First, we set

$$\Omega_{F_t} := \big(f(t), g(t), d(t), h(t), m(t)\big) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[[t]]^{\times 5}. \tag{3.47}$$

Then, comparing Eqs. (3.26), (3.30), (3.42) and (3.43) with Eqs. (2.2)–(2.5), respectively, and noting that, by Lemma 3.6, Eq. (2.1) is also satisfied by $f(t)$, we get the following theorem.

Theorem 3.14. *For any* $\alpha \geqslant 1$ *and* $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, *the* 5-*tuple* Ω_{F_t} *defined in Eq.* (3.47) *forms an* $\mathcal{N}CS$ *system over the differential operator* K-*algebra* $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

Furthermore, note that, the coefficients ψ_m ($m \geqslant 1$) are all K-derivations, hence are primitive elements of the Hopf algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ (see Eq. (3.4)). Then, by Theorem 2.5, we have the following correspondence between the NCSFs in the universal $\mathcal{N}CS$ system ($\mathcal{N}Sym$, Π) and the differential operators defined in this subsection.

Theorem 3.15. *For any* $\alpha \geqslant 1$ *and* $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ *as fixed before, there exists a unique* K-*Hopf algebra homomorphism* $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ *such that* $\mathcal{S}_{F_t}^{\times 5}(\Pi) = \Omega_{F_t}$.

More precisely, we have the following correspondence between the NCSFs in Π *and the differential operators in* Ω_{F_t}. *Namely, for any* $m \geqslant 1$, *we have*

$$\mathcal{S}_{F_t}(\Lambda_m) = \lambda_m, \tag{3.48}$$

$$\mathcal{S}_{F_t}(S_m) = s_m, \tag{3.49}$$

$$\mathcal{S}_{F_t}(\Psi_m) = \psi_m, \tag{3.50}$$

$$\mathcal{S}_{F_t}(\Phi_m) = \phi_m, \tag{3.51}$$

$$\mathcal{S}_{F_t}(\Xi_m) = \xi_m. \tag{3.52}$$

Note that, by Proposition 2.3, any one of Eqs. (3.48)–(3.52) in turn completely determines the homomorphism $\mathcal{S}_{F_t}$ in the theorem above.

One direct consequence of Theorem 3.15 above is that, for each $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we get a so-called *specialization* $\mathcal{S}_{F_t}$ (following the terminology in the theory of symmetric functions) of NCSFs by differential operators. From now on, we will call $\mathcal{S}_{F_t}$ the *differential operator specialization* of NCSFs associated with the automorphism $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$.

Finally, let us end this section with the following remark on a direct consequence of Theorem 3.15 to the study of the inversion problem [2,8].

Remark 3.16. By Lemma 3.12 and the comments after Eq. (3.41), when F_t has the form $F_t(z) = z - tH(z)$ with $H(z) \in K\langle\langle z\rangle\rangle^{\times n}$, the differential operators ψ_m's and ξ_m's (see Eqs. (3.39), (3.40) and (3.41)) becomes important for the study of the inversion problem [2,8]. On the other hand, as we pointed out earlier in Remark 2.7, by applying the homomorphism $\mathcal{S}_{F_t}$ and the correspondence in Theorem 3.15, we can transform the polynomial identities among the NCSFs [10,30] in the system Π into polynomial identities among the corresponding differential operators in the system Ω_{F_t}. Some of these identities may be used to study certain properties of the inverse map, the D-Log and the formal flow of $F_t(z)$. For more detailed study in this direction, see the forthcoming paper [30].

4. Differential operator specializations of NCSFs

Let $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$, $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{S}_{F_t}$ be as in the previous section. In this section, we study more properties and consequences of the differential operator specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) given in Theorem 3.15. First, we show in Proposition 4.1 that, for any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, the specialization $\mathcal{S}_{F_t}$ is a homomorphism of graded K-Hopf algebras from $\mathcal{N}Sym$ to the subalgebra $\mathcal{D}^{[\alpha]}\langle z\rangle := \mathcal{D}\langle z\rangle \cap \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, where $\mathcal{D}\langle z\rangle$ is the differential operator algebra of the polynomial algebra $K\langle z\rangle$, if and only if $F_t(z)=t^{-1}F(tz)$ for some automorphism $F(z)$ of $K\langle\langle z\rangle\rangle$. Consequently, for any F_t satisfying the condition above, by taking the graded duals, we get a graded K-Hopf algebra homomorphism $\mathcal{S}^*_{F_t}:\mathcal{D}^{[\alpha]}\langle z\rangle^*\to\mathcal{Q}Sym$ from the graded dual $\mathcal{D}^{[\alpha]}\langle z\rangle^*$ of $\mathcal{D}^{[\alpha]}\langle z\rangle$ to the Hopf algebra $\mathcal{Q}Sym$ [11,17,22] of quasi-symmetric functions (see Corollary 4.2). Secondly, we show in Theorem 4.3 that, with a properly defined group product $\circledast$ on the set $\mathbf{Hopf}(\mathcal{N}Sym,\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ of all K-Hopf algebra homomorphisms from $\mathcal{N}Sym$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, the correspondence of $F_t\in\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{S}_{F_t}\in\mathbf{Hopf}(\mathcal{N}Sym,\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ gives an isomorphism of groups. Finally, we show in Theorem 4.5 that the family of the specializations $\mathcal{S}_{F_t}$ (with all $n\geqslant 1$ and $F_t\in\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) can distinguish any two different NCSFs.

First let us fix the following notations.

Let $\mathcal{D}\langle z\rangle$ be the differential operator algebra of the polynomial algebra $K\langle z\rangle$, i.e. $\mathcal{D}\langle z\rangle$ is the unital subalgebra of $\mathrm{End}_K(K\langle z\rangle)$ generated by all K-derivations of $K\langle z\rangle$. For any $m\geqslant 0$, let $\mathcal{D}_{[m]}\langle z\rangle$ be the set of all differential operators U such that, for any homogeneous polynomial $h(z)\in K\langle z\rangle$ of degree $d\geqslant 0$, $Uh(z)$ either is zero or is homogeneous of degree $m+d$. For any $\alpha\geqslant 1$, set $\mathcal{D}^{[\alpha]}\langle z\rangle:=\mathcal{D}\langle z\rangle\cap\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Then, we have the grading

$$\mathcal{D}^{[\alpha]}\langle z\rangle=\bigoplus_{m\geqslant\alpha-1}\mathcal{D}_{[m]}\langle z\rangle, \tag{4.1}$$

with respect to which $\mathcal{D}^{[\alpha]}\langle z\rangle$ becomes a graded K-Hopf algebra.

Now, for any $\alpha\geqslant 2$, we let $\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ be the set of all automorphisms $F_t\in\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $F_t(z)=t^{-1}F(tz)$ for some automorphism $F(z)$ of $K\langle\langle z\rangle\rangle$. It is easy to check that $\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ is a subgroup of $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$.

Proposition 4.1. *For any $\alpha\geqslant 2$ and $F_t\in\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, the differential operator specialization $\mathcal{S}_{F_t}$ is a graded K-Hopf algebra homomorphism $\mathcal{S}_{F_t}:\mathcal{N}Sym\to\mathcal{D}^{[\alpha]}\langle z\rangle\subset\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ iff $F_t\in\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$.*

Proof. First, by the definition of $\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$, it is easy to see that $F_t(z)=z-H_t(z)\in\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ iff $H_t(z)$ can be written as

$$H_t(z)=\sum_{m\geqslant 1}t^m H_{[m]}(z) \tag{4.2}$$

with $H_{[m]}(z)$ ($m\geqslant 1$) homogeneous of degree $m+1$. Hence, we have $F_t=z-H_t(z)\in\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ iff the differential operator $[H_t(z)\frac{\partial}{\partial z}]$ can be written as

$$\left[H_t(z)\frac{\partial}{\partial z}\right]=\sum_{m\geqslant 1}t^m\delta_m \tag{4.3}$$

with $\delta_m=[H_{[m]}(z)\frac{\partial}{\partial z}]\in\mathcal{D}_{[m]}\langle z\rangle$ for all $m\geqslant 1$.

Now, we first assume that $\mathcal{S}_{F_t}$ in Theorem 3.15 is a graded K-Hopf algebra homomorphism from $\mathcal{N}Sym$ to $\mathcal{D}\langle z\rangle$, then, combining with Eq. (3.48), we have $\mathcal{S}_{F_t}(\Lambda_m)=\lambda_m\in\mathcal{D}_{[m]}\langle z\rangle$ for any $m\geqslant 1$. But, by Eq. (3.22) with $u(z)=z$, we have $F_t(z)=f(-t)\cdot z$, i.e. $H_t(z)=\sum_{m\geqslant 1}(-1)^{m-1}t^m\lambda_m z$ with $\lambda_m z$ homogeneous of degree $m+1$. Therefore Eq. (4.2) holds with $H_{[m]}(z)=(-1)^{m-1}\lambda_m z$ homogeneous of degree $m+1$ for any $m\geqslant 1$. Hence, we have $F_t\in\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$.

Conversely, assume $F_t(z) \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Then Eq. (4.3) holds with $\delta_m \in \mathcal{D}_{[m]}\langle z\rangle$ for all $m \geqslant 1$. Then by Eq. (3.24), it is easy to see that $\lambda_m \in \mathcal{D}_{[m]}\langle z\rangle$ for all $m \geqslant 1$. By Eq. (3.48) and the fact that $\mathcal{N}Sym$ is the free K-algebra generated by Λ_m $(m \geqslant 1)$, it is easy to see that $\mathcal{S}_{F_t}$ does preserve the gradings of $\mathcal{N}Sym$ and $\mathcal{D}\langle z\rangle$ defined by Eqs. (2.18) and (4.1), respectively. □

Now, for any $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ $(\alpha \geqslant 2)$, by the proposition above, we can take the graded dual of the graded K-Hopf algebra homomorphism $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle z\rangle$ and get the following corollary.

Corollary 4.2. *For any $\alpha \geqslant 2$ and $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$, let $\mathcal{D}^{[\alpha]}\langle z\rangle^*$ be the graded dual of the graded K-Hopf algebra $\mathcal{D}^{[\alpha]}\langle z\rangle$. Then,*

$$\mathcal{S}_{F_t}^* : \mathcal{D}^{[\alpha]}\langle z\rangle^* \to \mathcal{Q}Sym$$

is a homomorphism of graded K-Hopf algebras.

Next, let us consider the following observation. Note that, for any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, by looking at the differential operator specialization $\mathcal{S}_{F_t}$ in Theorem 3.15, we get the following map:

$$\begin{aligned} \mathbb{S} : \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle &\to \mathbf{Hopf}\big(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\big), \\ F_t &\to \mathcal{S}_{F_t}, \end{aligned} \tag{4.4}$$

where $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ denotes the set of K-Hopf algebra homomorphisms from $\mathcal{N}Sym$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

We claim the map $\mathbb{S}$ above is a bijection. To see it is injective, first note that $\mathcal{S}_{F_t}$ is uniquely determined by $f(t) = \mathcal{S}_{F_t}(\lambda(t))$ since $\mathcal{N}Sym$ is freely generated by the coefficients Λ_m $(m \geqslant 1)$ of $\lambda(t)$. While by Eq. (3.22) with $u(z) = z$, we have $F_t(z) = f(-t) \cdot z$. Therefore, $f(t)$ conversely completely determines F_t itself. Hence $\mathbb{S}$ is injective. To show the surjectivity of $\mathbb{S}$, let $\mathcal{T}$ be any element of $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$. Then $\mathcal{T}(\Phi(t))$ must have all its coefficients primitive in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. This is because the NCSFs Φ_m $(m \geqslant 1)$ are all primitive and any Hopf algebra homomorphism preserves primitive elements. Since the only primitive elements of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ are K-derivations, we have $\mathcal{T}(\Phi(t)) \in t\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!] = t\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$. Then by Proposition 3.11, there exists a unique $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that its D-Log $a_t(z)$ is given by the property $[a_t(z)\frac{\partial}{\partial z}] = -\mathcal{T}(\Phi(t))$. By Eq. (3.51) in Theorem 3.15, we have $\mathcal{S}_{F_t}(\Phi(t)) = \mathcal{T}(\Phi(t))$, hence, we must have $\mathcal{S}_{F_t} = \mathcal{T}$ since, by Proposition 2.3, $\mathcal{N}Sym$ is also freely generated by Φ_m $(m \geqslant 1)$. So $\mathbb{S}$ is also surjective.

Next, we consider the following group product on the set $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$, with which the bijection $\mathbb{S}$ in Eq. (4.4) becomes an isomorphism of groups. Note that the convolution product $*$ of the linear maps from $\mathcal{N}Sym$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ is not a right choice, for $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ is not closed under the convolution product.

First, let $\mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ be the set of all sequences of divided powers $\{a_m \mid m \geqslant 0\}$ (with $a_0 = 1$) of the Hopf algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Consider the following map:

$$\begin{aligned} \mathbb{D} : \mathbf{Hopf}\big(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\big) &\to \mathbf{DPS}\big(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\big), \\ \mathcal{T} &\to \big\{\mathcal{T}(S_m) \bigm| m \geqslant 0\big\}. \end{aligned} \tag{4.5}$$

Note that the map $\mathbb{D}$ is well defined for the NCSFs, $\{S_m \mid m \geqslant 0\}$ form a sequence of divided powers of $\mathcal{N}Sym$ and any Hopf algebra homomorphism preserves sequences of divided powers.

We claim the map $\mathbb{D}$ above is a bijection. The injectivity is obvious for $\mathcal{N}Sym$ is the free K-algebra generated by S_m $(m \geqslant 1)$ (see Proposition 2.3). To see the surjectivity of $\mathbb{D}$, let $\{a_m \mid m \geqslant 0\} \in \mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ and $\mathcal{A} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ be the unique K-algebra homomorphism that maps S_m to a_m for any $m \geqslant 1$. Then, by applying Theorem 2.5(b) to the $\mathcal{N}$CS system $\mathcal{A}^{\times 5}(\Pi)$ over $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, we see that $\mathcal{A}$ must also be a K-Hopf algebra homomorphism, i.e. $\mathcal{A} \in \mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$. By the definition of $\mathcal{A}$, we have $\mathbb{D}(\mathcal{A}) = \{a_m \mid m \geqslant 0\}$. Hence $\mathbb{D}$ is also surjective.

Now we identify any sequence of divided powers $\{a_m \mid m \geqslant 0\}$ of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ with its generating function $a(t) := \sum_{m \geqslant 0} a_m t^m$. Note that, in general, a sequence $\{a_m \mid m \geqslant 0\}$ with $(a_0 = 1)$ is a divide series iff its generating function $a(t)$ satisfies $\Delta a(t) = a(t) \otimes a(t)$ (and $a_0 = 1$). Therefore we can view $\mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ as the subset of the elements $a(t)$ of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$ such that $\Delta a(t) = a(t) \otimes a(t)$ (and $a_0 = 1$). Note that, for any $a(t), b(t) \in \mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle) \subset \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$, we have

$$\begin{aligned}
\Delta\big(a(t)b(t)\big) &= \Delta\big(a(t)\big)\Delta\big(b(t)\big) \\
&= \big(a(t) \otimes a(t)\big)\big(b(t) \otimes b(t)\big) \\
&= \big(a(t)b(t)\big) \otimes \big(a(t)b(t)\big).
\end{aligned}$$

So, $\mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ is closed under the algebra product of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$. Since each $a(t) \in \mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ is also invertible in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$, for $a(0) = 1$, $\mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ with the algebra product of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[\![t]\!]$ forms a group. By identifying the set $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ with $\mathbf{DPS}(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ via the bijection $\mathbb{D}$ in Eq. (4.5), we get a group product denoted by $\circledast$ for the set $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$. Let $\circ$ denote the group product of $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ from the composition of automorphisms. Then we have the following theorem.

Theorem 4.3. *For any $\alpha \geqslant 1$, the map*

$$\begin{aligned}
\mathbb{S} : \big(\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle, \circ\big) &\to \big(\mathbf{Hopf}\big(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\big), \circledast\big), \\
F_t &\to \mathcal{S}_{F_t},
\end{aligned} \tag{4.6}$$

is an isomorphism of groups.

Proof. By the discussion after Eq. (4.4), we only need to show the map $\mathbb{S}$ is a homomorphism of groups. Furthermore, by the definition of the group product $\circledast$ above, it will be enough to show that, for any $U_t(z), V_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$,

$$\mathcal{S}_{U_t \circ V_t}\big(\sigma(t)\big) = \mathcal{S}_{U_t}\big(\sigma(t)\big)\mathcal{S}_{V_t}\big(\sigma(t)\big), \tag{4.7}$$

$$\mathcal{S}_{U_t^{-1}}\big(\sigma(t)\big) = \mathcal{S}_{U_t}\big(\sigma(t)\big)^{-1}, \tag{4.8}$$

where, as before, $\sigma(t) = \sum_{m \geqslant 0} S_m t^m$ is the generating function of the complete elementary homogeneous NCSFs $\{S_m \mid m \geqslant 0\}$.

First, by Theorem 3.15, we know that, for any $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, $\mathcal{S}_{F_t}(\sigma(t)) = g(t)$, which is the differential operator defined in Eq. (3.21). Therefore, for any $v_t(z) \in K[\![t]\!]\langle\langle z\rangle\rangle$, by Eq. (3.23), we have

$$\mathcal{S}_{F_t}\big(\sigma(t)\big)v_t(z) = v_t\big(F_t^{-1}(z)\big). \tag{4.9}$$

In particular, for any $u(z) \in K\langle\langle z\rangle\rangle$, we have

$$\begin{aligned}
\mathcal{S}_{U_t}\big(\sigma(t)\big)\mathcal{S}_{V_t}\big(\sigma(t)\big)u(z) &= \mathcal{S}_{U_t}\big(\sigma(t)\big)u\big(V_t^{-1}(z)\big) \\
&= u\big(V_t^{-1} \circ U_t^{-1}(z)\big) \\
&= u\big((U_t \circ V_t)^{-1}(z)\big) \\
&= \mathcal{S}_{U_t \circ V_t}\big(\sigma(t)\big)u(z).
\end{aligned}$$

Hence, we have Eq. (4.7).

To show Eq. (4.8), applying Eq. (4.9) to $F_t = U_t^{-1}$ and $v_t(z) = u(z)$ for any $u(z) \in K\langle\langle z\rangle\rangle$, we have,

$$\mathcal{S}_{U_t^{-1}}\big(\sigma(t)\big)u(z) = u\big(U_t(z)\big). \tag{4.10}$$

On the other hand, by Theorem 3.15 and Eq. (3.22), we also have

$$\begin{aligned}
\mathcal{S}_{U_t}\big(\sigma(t)\big)^{-1}u(z) &= \mathcal{S}_{U_t}\big(\sigma(t)^{-1}\big)u(z) \\
&= \mathcal{S}_{U_t}\big(\lambda(-t)\big)u(z) \\
&= f(-t)u(z) \\
&= u\big(U_t(z)\big).
\end{aligned}$$

Hence, we have Eq. (4.8). □

Remark 4.4. Note that one implication of Theorem 4.3 and Proposition 4.1 is that all specializations $\mathcal{T} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ of NCSFs, which are also K-Hopf algebra homomorphisms, are exactly the specializations $\mathcal{S}_{F_t}$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$); while all specializations $\mathcal{T} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle z\rangle$ of NCSFs, which are also graded K-Hopf algebra homomorphisms, are exactly the specializations $\mathcal{S}_{F_t}$ with $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$.

Next we show that, for any $\alpha \geqslant 2$, the family of the differential operator specializations $\mathcal{S}_{F_t}$ (with all $n \geqslant 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) can distinguish any two different NCSFs. This statement obviously is the same as the following theorem.

Theorem 4.5. *In both commutative and noncommutative cases, the following statement holds.*

For any fixed $\alpha \geqslant 2$ and non-zero $P \in \mathcal{N}Sym$, there exist $n \geqslant 1$ (the number of the free variable z_i's) and $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $\mathcal{S}_{F_t}(P) \neq 0$.

This is probably the only result derived in this paper for which the commutative case does not follow from the noncommutative case by the procedure of abelianization. Instead, as one can easily show the noncommutative case actually follows from the commutative case by choosing any lifting of F_t in commutative variables to an automorphism of formal power series algebra in noncommutative variables. Therefore, we only need to prove the theorem for the commutative case.

Proof. First, by Proposition 2.3, we may view $\mathcal{N}Sym$ as the free algebra $K\langle\Phi\rangle$ generated by Φ_m ($m \geqslant 1$). Below we will write $\mathcal{N}Sym$ as $K\langle\Phi\rangle$ and view any NCSF $P \in \mathcal{N}Sym = K\langle\Phi\rangle$ as a polynomial $P(\Phi)$ in Φ_m's. Secondly, let $\mathcal{L}(\Phi) \subset K\langle\Phi\rangle$ be the free Lie algebra generated by Φ_m ($m \geqslant 1$). Then, for any free variables z, by a similar argument as the proof of the bijectivity of the map $\mathbb{D}$ in Eq. (4.5), it is easy to see that, via the restriction and extension of the homomorphisms, the set $\mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle)$ is in 1–1 correspondence with the set $\mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$ of the Lie algebra homomorphisms from $\mathcal{L}(\Phi)$ to $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$. Combining with the isomorphism $\mathbb{S}$: $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle \simeq \mathbf{Hopf}(\mathcal{N}Sym, \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$ in Theorem 4.3, we see that the set of the specializations $\mathcal{S}_{F_t}$ with $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ is in 1–1 correspondence with the set $\mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$.

Now let $\mathcal{K}$ be the set of all NCSFs $P(\Phi) \in K\langle\Phi\rangle$ that are mapped to zero by the specialization $\mathcal{S}_{F_t}$ for any number $n \geqslant 1$ of free variables $z = (z_1, z_2, \ldots, z_n)$ and any $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. By the 1–1 correspondence discussed above, it is easy to see that $P(\Phi) \in \mathcal{K}$ iff $P(\Phi)$ satisfies the following property:

(**K**) *For any number $n \geqslant 1$ of free variables $z = (z_1, z_2, \ldots, z_n)$ and any Lie algebra homomorphism $\sigma : \mathcal{L}(\Phi) \to \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$, if we still denote by σ the extended K-algebra homomorphism from $K\langle\Phi\rangle$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, then we have $\sigma(P(\Phi)) = 0$.*

Now we assume that the theorem is false, i.e. $\mathcal{K} \neq 0$ and derive a contradiction as follows.

Claim 1. *$\mathcal{K}$ is invariant under linear transformations. More precisely, for any integers $M, N \geqslant 1$ and $a_{i,j} \in K$ with $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$, let $Y := \{Y_m \mid m \geqslant 1\}$, where*

$$Y_m := \begin{cases} \sum_{j=1}^{N} a_{m,j}\Phi_j & \text{if } 1 \leqslant m \leqslant M, \\ \Phi_m & \text{if } m > M. \end{cases} \tag{4.11}$$

Then, for any $P(\Phi) \in \mathcal{K}$, we have $P(Y) \in \mathcal{K}$.

Proof. It will be enough to show that $P(Y)$ satisfies the property (**K**) when $P(\Phi)$ does. For any $\sigma \in \mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$, we define a homomorphism $\eta : \mathcal{L}(\Phi) \to \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ of Lie algebras by setting $\eta(\Phi_m) = \sigma(Y_m)$ for any $m \geqslant 1$. Then it is easy to see that $\sigma(P(Y)) = \eta(P(\Phi))$. Hence, when $P(\Phi)$ satisfies the property (**K**), we have $\sigma(P(Y)) = \eta(P(\Phi)) = 0$ for any $\sigma \in \mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$. □

Claim 2. *Let $\mathcal{H}$ be the set of all elements $Q(\Phi) \in \mathcal{K}$ such that $Q(\Phi)$ is homogeneous in each Φ_m that is involved in $Q(\Phi)$. Then $\mathcal{H} \neq 0$.*

Proof. Let $P(\Phi)$ be any non-zero element of $\mathcal{K}$. By Claim 1 above, we may assume that $P(\Phi)$ involves exactly Φ_m ($1 \leqslant m \leqslant N$) for some $N \geqslant 1$. Let $y = (y_1, y_2, \ldots, y_N)$ be N formal central parameters, i.e. they commute with each other, and also with Φ_m's and all free variables z_i's under the consideration. Let $P(\Phi; y)$ be the polynomial in Φ and y obtained by replacing Φ_m $(1 \leqslant m \leqslant N)$ in $P(\Phi)$ by $y_m\Phi_m$. We view $P(\Phi; y)$ as a polynomial in y with coefficients in $K\langle\Phi\rangle$ and write it as

$$P(\Phi; y) = \sum_{I \in \mathbb{N}^N} P_I(\Phi) y^I. \tag{4.12}$$

Now, for any $\vec{v} \in K^{\times N}$, by Claim 1, we have $P(\Phi; \vec{v}) \in \mathcal{K}$. Therefore, $\sigma(P(\Phi; \vec{v})) = 0$ for any $\sigma \in \mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$. In particular, for any $u(z) \in K\langle\langle z\rangle\rangle$, we have $\sigma(P(\Phi; \vec{v})) \cdot u(z) = 0$. Therefore, when we write $\sigma(P(\Phi; y)) \cdot u(z) = \sum_{I \in \mathbb{N}^N} y^I \sigma(P_I(\Phi)) \cdot u(z)$ as a formal power series in z with coefficients in $K[y]$, all its coefficients will vanish at any $\vec{v} \in K^{\times N}$. Hence, as polynomials in the commutative variables y, these coefficients must be identically zero, since K^N is obviously dense with respect to the Zariski topology of K^N. Therefore, for any $u(z) \in K\langle\langle z\rangle\rangle$, $\sigma(P(\Phi; y))u(z) = 0$ as an element of $K[y]\langle\langle z\rangle\rangle$. Hence, $\sigma(P(\Phi; y)) = 0$ as a polynomial of y with coefficients in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. In particular, as elements of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, all the coefficients $\sigma(P_I(\Phi))$ of y^I in $\sigma(P(\Phi; y))$ are equal to zero. Since this is true for any $\sigma \in \mathbf{Lie}(\mathcal{L}(\Phi), \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$, all the coefficients $P_I(\Phi)$ of y^I in Eq. (4.12) are in $\mathcal{K}$.

On the other hand, by the definition of $P(\Phi; y)$ above, it is easy to see that, for any fixed $I = (i_1, \dots, i_N) \in \mathbb{N}^N$, $P_I(\Phi)$ in Eq. (4.12) is homogeneous in each Φ_m $(1 \leqslant m \leqslant N)$ of partial degree i_m. Therefore, all $P_I(\Phi) \in \mathcal{H}$. Since not all $P_I(\Phi)$ can be zero, otherwise $P(\Phi)$ would be zero, hence Claim 2 follows. □

Claim 3. *Let $\mathcal{H}_1$ be the set of all elements $Q(\Phi) \in \mathcal{K}$ such that $Q(\Phi)$ is homogeneous of partial degree* 1 *in each Φ_m that is involved in $Q(\Phi)$. Then $\mathcal{H}_1 \neq 0$.*

Proof. By using Claim 2, we first fix a non-zero element $P(\Phi) \in \mathcal{H}$ and, by Claim 1, we assume that $P(\Phi)$ involves exactly Φ_m $(1 \leqslant m \leqslant N)$ for some $N \geqslant 1$. For each $1 \leqslant m \leqslant N$, we assume that $P(\Phi)$ is homogeneous of degree $d_m \geqslant 1$ in Φ_m. Hence $P(\Phi)$ is homogeneous of total degree $d := \sum_{1 \leqslant m \leqslant N} d_m$.

Let $B = \{(m, j) \mid 1 \leqslant m \leqslant N;\ 1 \leqslant j \leqslant d_m\}$ and $W = \{W_{m,j} \mid (m, j) \in B\}$ be any subset of $\{\Phi_m \mid m \geqslant 1\}$ with $|W| = d$. Let $y = \{y_{m,j} \mid (m, j) \in B\}$ be a family of central formal parameters. For any $1 \leqslant m \leqslant N$, set $Y_m := \sum_{j=1}^{d_m} y_{m,j} W_{m,j}$ and $Y := \{Y_m \mid 1 \leqslant m \leqslant N\}$. Let $\tilde{P}(W, y) := P(Y)$. We view $\tilde{P}(W, y)$ as a polynomial in y with coefficients in $K\langle W\rangle$ and let $Q(W)$ be the coefficient of the monomial $\prod_{(m,j)\in B} y_{m,j}$ in $\tilde{P}(W, y)$. First, by a similar argument as in the proof of Claim 2 above, we see that $Q(W)$ as well as other coefficients of the monomials of y appearing in $\tilde{P}(W, y)$ are in $\mathcal{K}$. Secondly, it is easy to see that $Q(W)$ is homogeneous of partial degree 1 in each $W_{m,j}$ $((m, j) \in B)$. Thirdly, $Q(W) \neq 0$ since $P(\Phi)$ can be recovered from $Q(W)$ by replacing $W_{m,j}$ by Φ_m for any $(m, j) \in B$ and then dividing the multiplicity factor $d_1!d_2!\cdots d_m!$. Therefore, $\tilde{Q}(W)$ is a non-zero element of $\mathcal{H}_1$ and Claim 3 follows. □

Finally, we derive a contradiction as follows.

Let $P(\Phi)$ be a non-zero element of $\mathcal{H}_1$ with the least total degree in Φ among all non-zero elements of $\mathcal{H}_1$. By Claim 1, we may assume that $P(\Phi)$ is homogeneous of partial degree 1 in each Φ_m $(1 \leqslant m \leqslant N)$ for some $N \geqslant 1$ and does not depend on any Φ_m with $m > N$. Now we write $P(\Phi)$ as

$$P(\Phi)=\sum_{m=1}^{N} P_m(\Phi)\Phi_m. \tag{4.13}$$

First, not all $P_m(\Phi)$'s above can be zero since $P(\Phi)\neq 0$. Without losing any generality, let us assume $P_N(\Phi)\neq 0$. Secondly, $P_N(\Phi)$ is homogeneous of partial degree 1 in each Φ_m ($1\leqslant m\leqslant N-1$) and does not depend on any Φ_m with $m\geqslant N$. Thirdly, the total degree $\deg P_N(\Phi)=\deg P(\Phi)-1<\deg P(\Phi)$.

Now, for any $n\geqslant 1$, let $z=(z_1,z_2,\ldots,z_n)$ be n free variables and w a free variable that is independent with z. For any $u(z)\in K\langle\langle z\rangle\rangle$ with $o(u(z))\geqslant\alpha$ and any $\sigma\in\mathbf{Lie}(\mathcal{L}(\Phi),\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$, let $\tilde{\sigma}:\mathcal{L}(\Phi)\to\mathcal{D}er^{[\alpha]}\langle\langle x;w\rangle\rangle$ be the unique Lie algebra homomorphism which maps Φ_m ($1\leqslant m\leqslant N-1$) to $\sigma(\Phi_m)$; Φ_N to $[u(z)\frac{\partial}{\partial w}]$ and Φ_m ($m>N$) to zero. Then, we have $\tilde{\sigma}(P_N(\Phi))=\sigma(P_N(\Phi))$ and, for any $1\leqslant m\leqslant N-1$, $\tilde{\sigma}(\Phi_m)\cdot w=\sigma(\Phi_m)\cdot w=0$ for $\sigma(\Phi_m)\in\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$. By the fact $P(\Phi)\in\mathcal{K}$ and Eq. (4.13), we have

$$\begin{aligned}
0&=\tilde{\sigma}\big(P(\Phi)\big)\cdot w\\
&=\sigma\big(P_N(\Phi)\big)\left[u(z)\frac{\partial}{\partial w}\right]\cdot w\\
&=\sigma\big(P_N(\Phi)\big)u(z).
\end{aligned}$$

Since the differential operator $\sigma(P_N(\Phi))$ annihilates any $u(z)\in K\langle\langle z\rangle\rangle$ with $o(u(z))\geqslant\alpha$, by Lemma 3.5, $\sigma(P_N(\Phi))=0$. Since this is true for any $\sigma\in\mathbf{Lie}(\mathcal{L}(\Phi),\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle)$, hence, $P_N(\Phi)\in\mathcal{K}$. By the facts pointed in the previous paragraph, we further have $P_N(\Phi)\in\mathcal{H}_1$. But the total degree $\deg P_N(\Phi)<\deg P(\Phi)$, which contradicts to the choice of $P(\Phi)$. $\square$

Finally, let us end this paper with the following remarks.

First, it is easy to see that, for any $\alpha\geqslant 2$, all the results derived in this paper still hold if the base algebra $K[\![t]\!]$ is replaced by $K[t]$. Secondly, in the forthcoming paper [32], by using Theorem 4.5 above and some connections of the $\mathcal{NCS}$ system $(\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle,\Omega_{F_t})$ with the $\mathcal{NCS}$ systems constructed in [32] over the Grossman–Larson Hopf algebra of labeled rooted trees, the following much stronger version of Theorem 4.5 will be proved.

Let $\mathbb{B}_t^{[\alpha]}\langle z\rangle$ be the set of automorphisms $F_t(z)=z-H_t(z)$ of the polynomial algebra $K[t]\langle z\rangle$ over $K[t]$ such that the following conditions are satisfied.

- $H_{t=0}(z)=0$.
- $H_t(z)$ is homogeneous in z of degree $d\geqslant\alpha$.
- With a proper permutation of the free variables z_i's, the Jacobian matrix $JH_t(z)$ is strictly lower triangular.

Theorem 4.6. *(See [32].) In both commutative and noncommutative cases, the following statement holds.*

For any fixed $\alpha\geqslant 2$ and non-zero $P\in\mathcal{N}Sym$, there exist $n\geqslant 1$ (the number of the free variable z_i's) and $F_t(z)\in\mathbb{B}_t^{[\alpha]}\langle z\rangle$ such that $\mathcal{S}_{F_t}(P)\neq 0$.

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